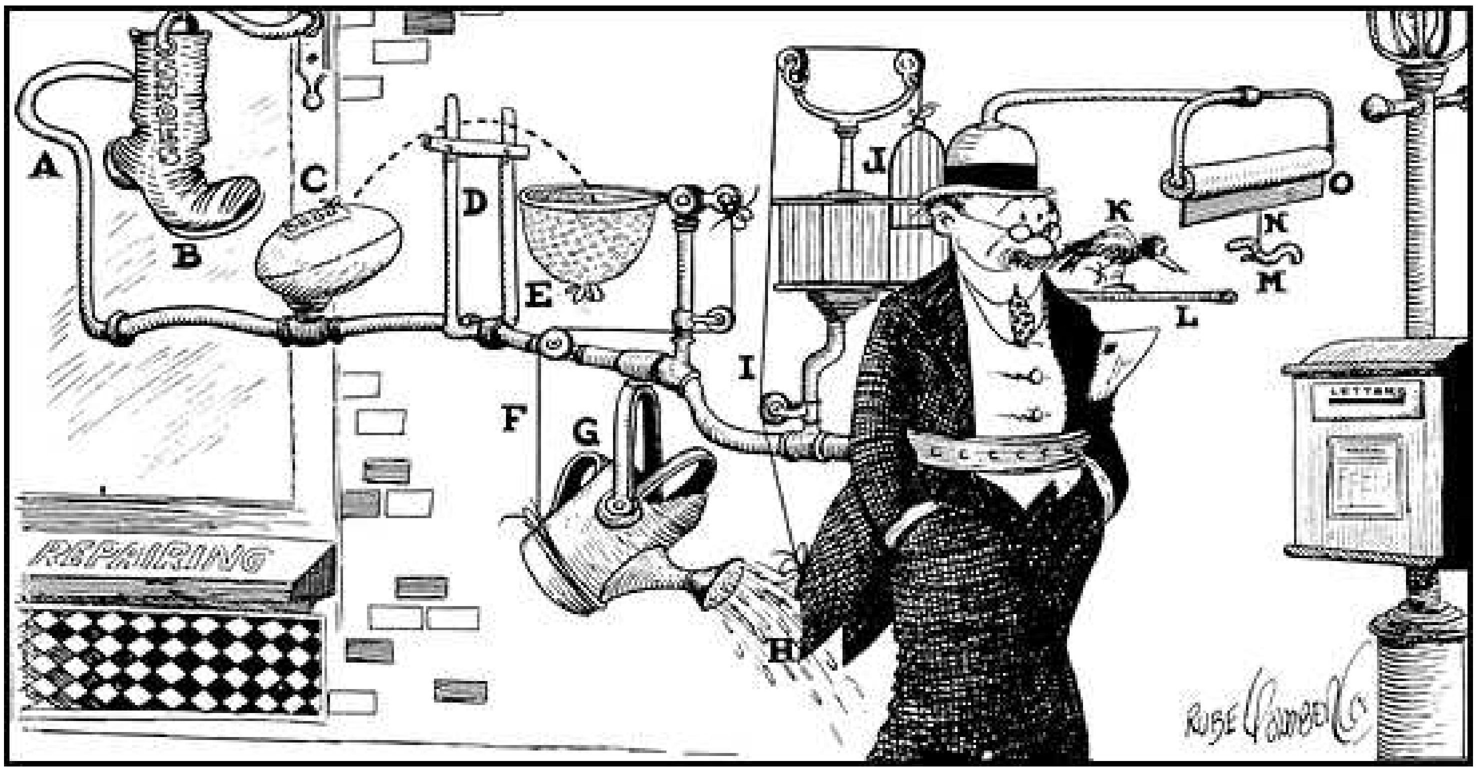
Corrections to static antenna patterns in the kHz range

Malik Rakhmanov



Rube Goldberg

(A comically involved, complicated invention, laboriously contrived to perform a simple operation)

Things one can find in DCC

LIGO documents on the frequency dependence of the LIGO antenna patterns and the implication for calibration:

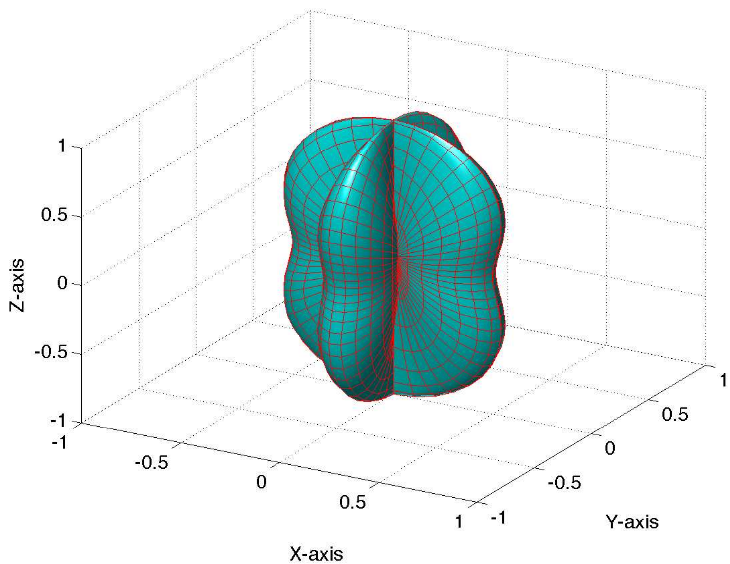
- T970101-B, D. Sigg, *Strain calibration in LIGO*,
- T030296, D. Sigg and R. Savage, *Analysis proposal to search for gravitational waves at multiples of the LIGO arm cavity free-spectral-range frequency*,
- T030186, J. Markowicz, R.L. Savage, and P. Schwinberg, *Development of a readout scheme for high-frequency gravitational waves*,
- G050205, M. Rakhmanov and R. Savage, *LIGO detector response at high frequencies and its implications for calibration above 1kHz*,
- T050136, Hunter Elliott, *Analysis of the frequency dependence of the LIGO directional sensitivity (Antenna Pattern) and implications for detector calibration*,
- G060665, M. Rakhmanov, *Frequency corrections to antenna-patterns: forward detector transfer function*,
- G060667, R. Savage et al., *LIGO high-frequency response to length- and GW-induced optical path length variations*,
- T070037, Jeffrey Parker, *Development of a high-frequency burst analysis pipeline.*

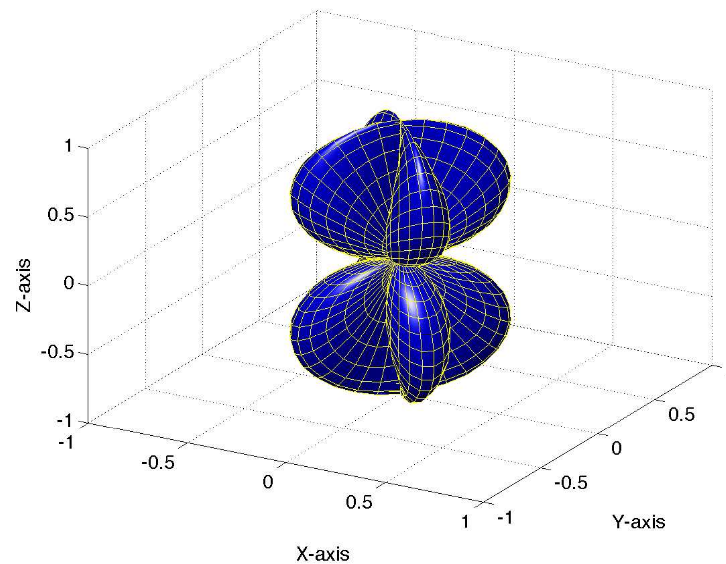
I've seen this before.

Antenna patterns at $f = 0$ or DC:

$F_+(\phi, \theta)$ – response to +polarization,

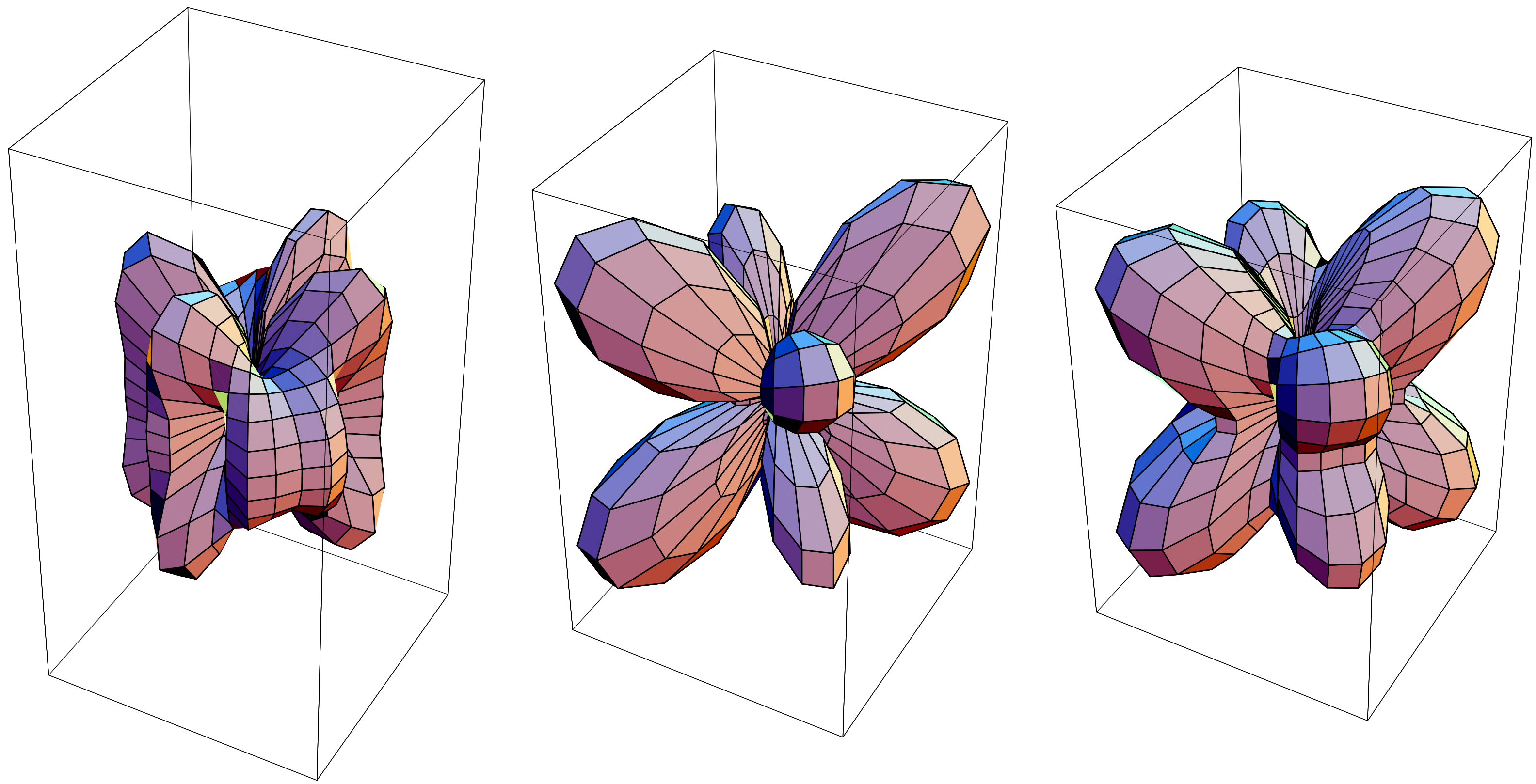
$F_\times(\phi, \theta)$ – response to $\times$polarization. [* Is there something like $F(\phi, \theta, \psi)$ *] ?





Long time ago ...

Antenna patterns at f =FSR: $H_+(f|\phi,\theta)$, $H_\times(f|\phi,\theta)$, $H_{\text{ave}}(f|\phi,\theta)$.



T970101-B, D.Sigg, *Strain calibration in LIGO*.

$H_{gw}(s)$: it's as simple as one, two, three...

Source location: (ϕ, θ), polarization angle: ψ, unit vector $\vec{n}$, and rotation matrix R:

$$\begin{aligned} n_x &= \sin\theta\cos\phi \\ n_y &= \sin\theta\sin\phi \ , \\ n_z &= \cos\theta \end{aligned} \qquad E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \begin{aligned} R &= R_z(\psi) R_y(\theta) R_z(\phi), \\ E' &= R^T \, E \, R \end{aligned}$$

Introduce the equivalent phase response and the Fabry-Perot cavity field response ($s = 2\pi i f$):

$$\phi_i = \frac{A_i - B_i e^{-2sT}}{2sT}, \qquad H_{\rm FP} = \frac{1 - r_a r_b}{1 - r_a r_b e^{-2sT}},$$

where r_a, r_b are mirror reflectivities and A_i and B_i $(i = x, y)$ are phase factors:

$$A_i = \frac{1 - e^{-(1-n_i)sT}}{1 - n_i}, \qquad B_i = \frac{1 - e^{-(1+n_i)sT}}{1 + n_i}.$$

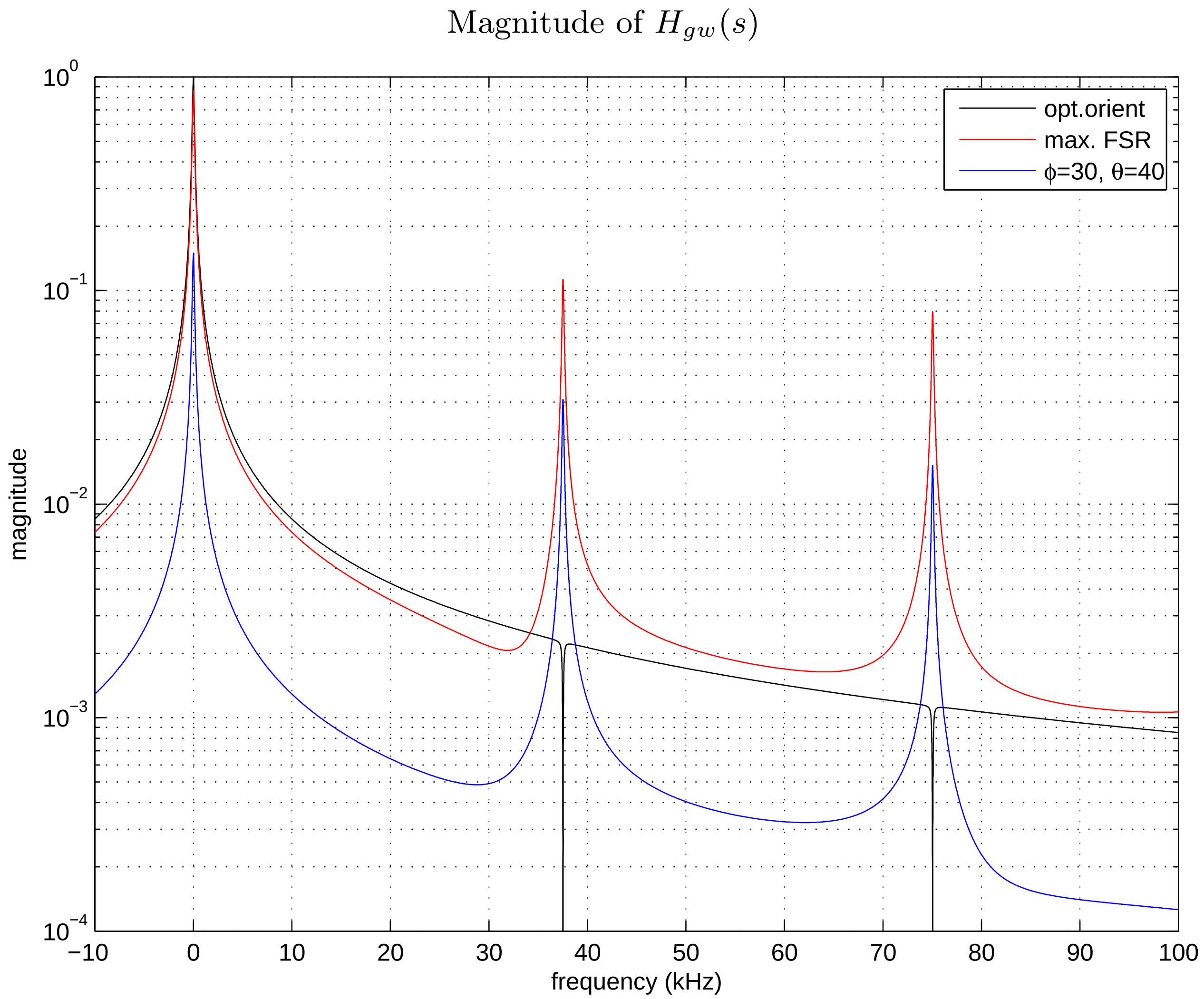
Then the response to gravitational waves is

$$H_{gw} = \frac{1}{2} \, H_{\rm FP} \, \left(E'_{xx} \, \phi_x - E'_{yy} \, \phi_y \right).$$

For a polarized source, we can also introduce the redundant responses:

$$H_+ = H_{gw}|_{\psi=0}\,, \qquad H_\times = H_{gw}|_{\psi=45^\circ}\,, \qquad H_{\rm circ} = H_+ + i H_\times .$$

Sometimes up and sometimes down



Back to the future

Collect all angles: $\Omega = (\phi, \theta, \psi)$.

Detector response = convolution:

$$x(t) = \int_0^T \left[H_+(t-t', \Omega)\; h_+(t') + H_\times(t-t', \Omega)\; h_\times(t')\right] dt'.$$

In Fourier domain:

$$\tilde{x}(f) = H_+(f, \Omega)\; \tilde{h}_+(f) + H_\times(f, \Omega)\; \tilde{h}_\times(f).$$

At low frequencies ($f \ll \mathrm{FSR}$):

$$H_{+,\times}(f, \Omega) \approx F_{+,\times}(\Omega) * H_{\text{pole}}(f),$$

where $F_{+,\times}(\Omega)$ are static antenna-patterns.

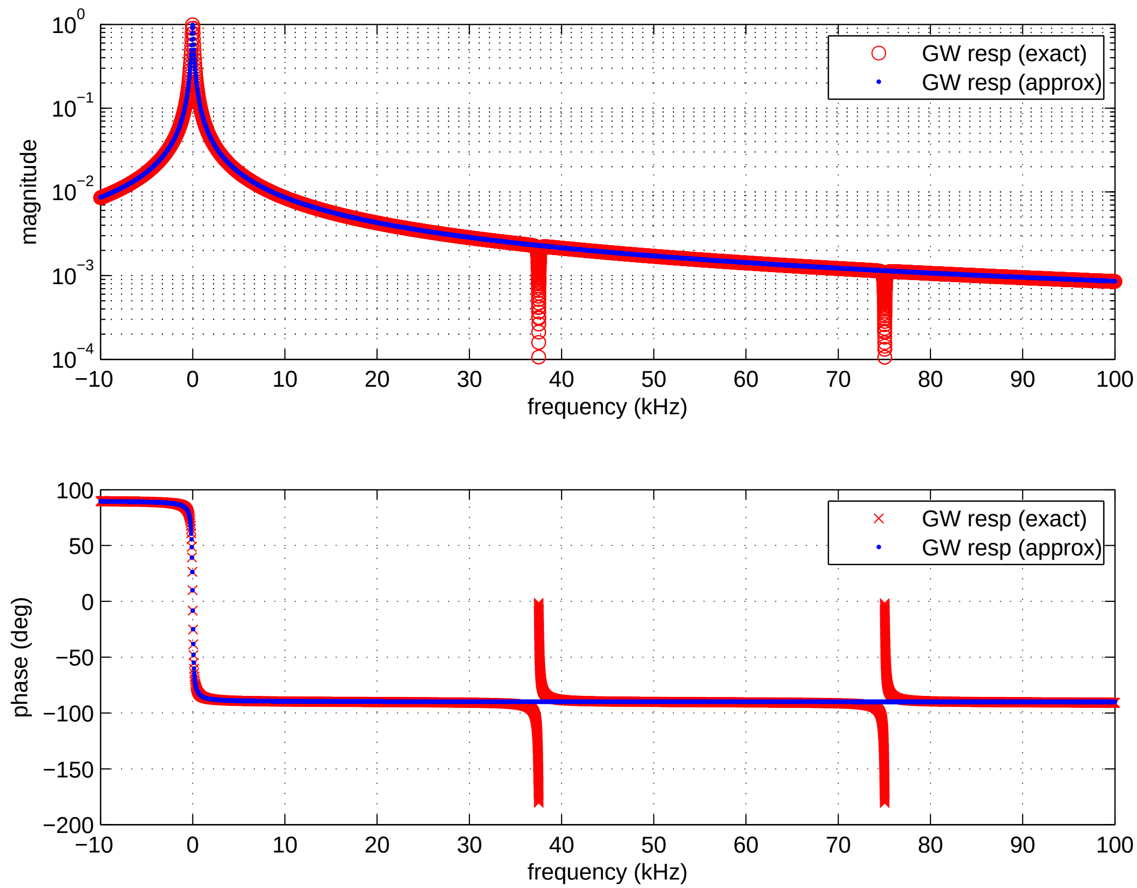
$H_{\text{pole}}(f)$ is an approximate frequency response for optimal orientation:

$$H_{\text{pole}}(f) = \frac{1}{1 + if/f_{cav}}, \qquad\qquad f_{cav} \approx 86 \;\; \text{Hz}.$$

This is called the long-wavelength approximation.

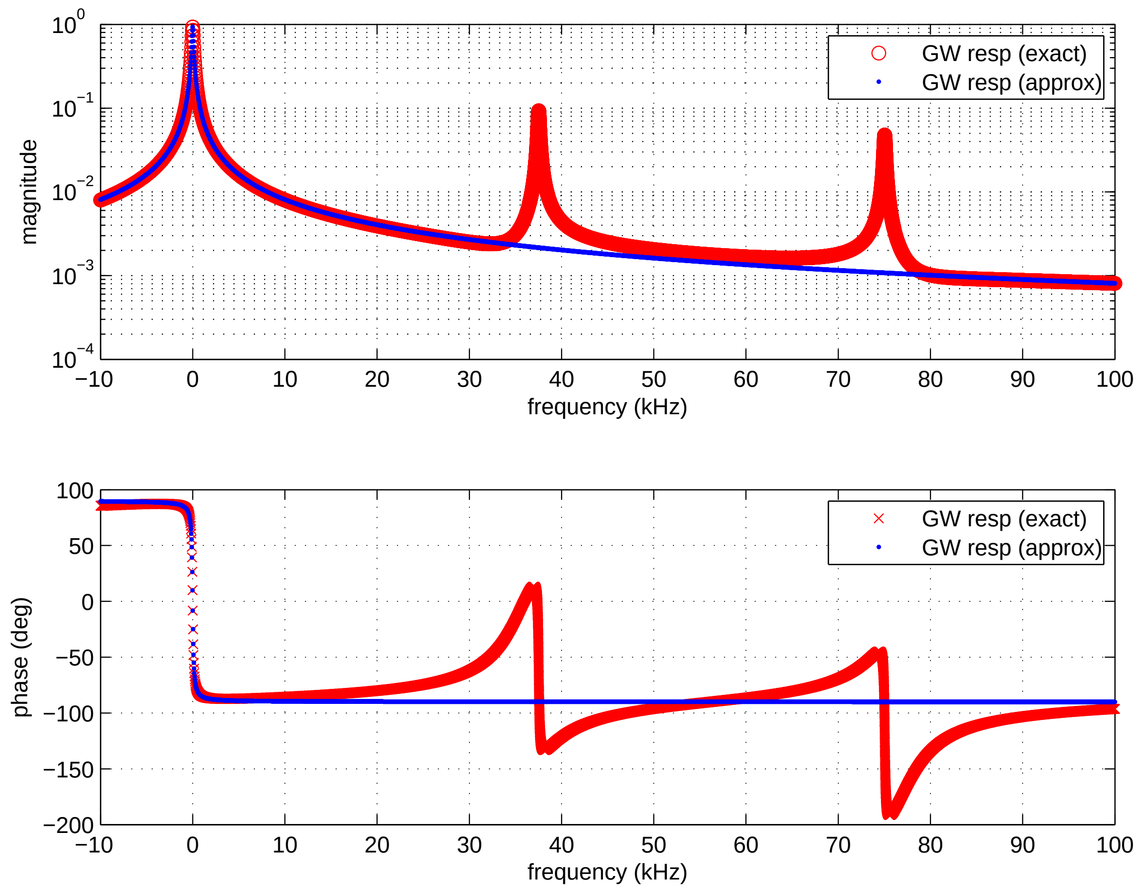
Fell right on it!

$H_{gw}(s)$: exact and approximate forms. (source coordinates: $\phi = 0$, $\theta = 0$, $\psi = 0$).



Must be a coincidence...

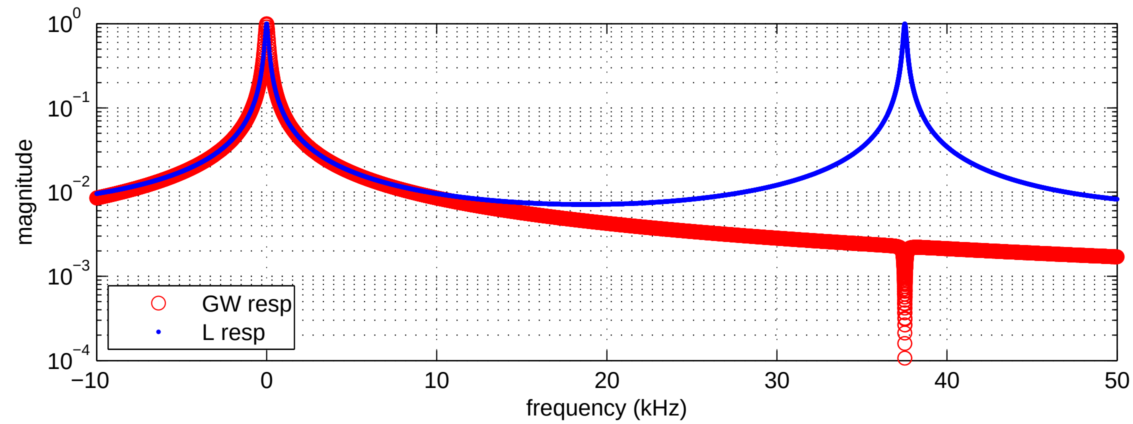
$H_{gw}(s)$: exact and approximate forms. (source coordinates: $\phi = 0$, $\theta = 20°$, $\psi = 0$).

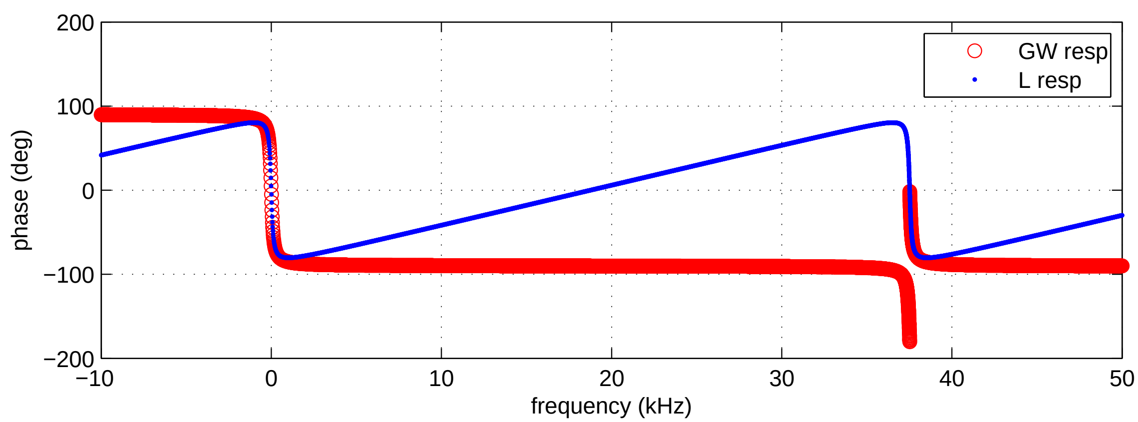


Apples and oranges

Comparison of $H_{gw}(s)$ and $H_L(s)$.

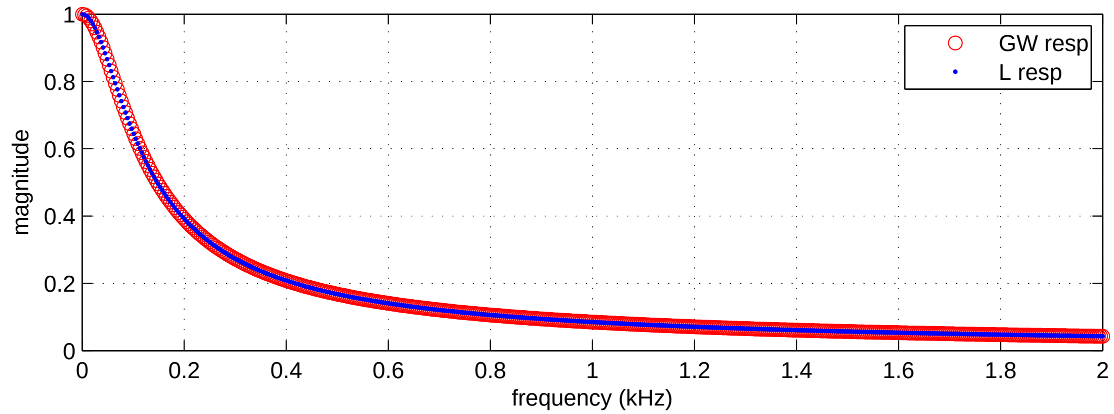
Calibration measures $H_L(f)$ and not $H_{gw}(f)$.

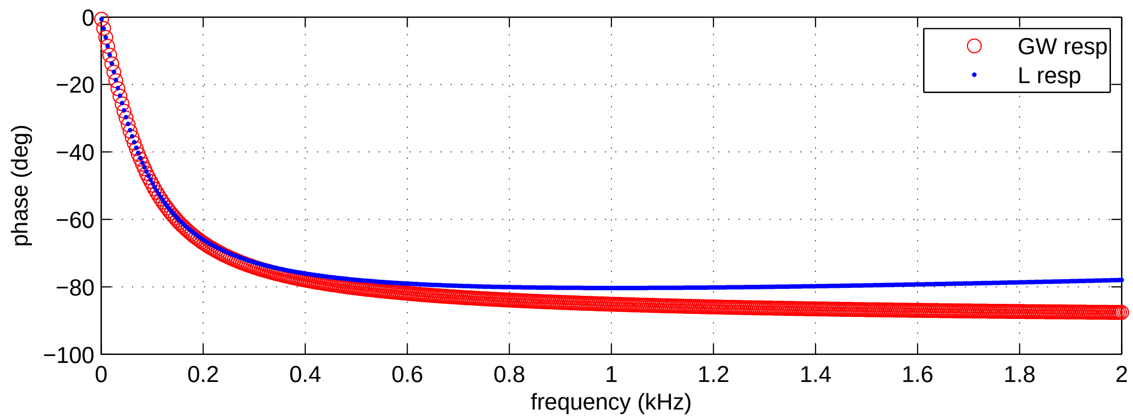




Tiny details

At low frequencies the magnitude of the length response and that of the gravitational-wave response are almost the same. The phase is slightly different though.





Multitude of species

Complete response contains three effects.

	geometric factor	photon transit time	Fabry-Perot effect
approx.	$F_{+,\times}(\phi,\theta)$	1	$H_{\rm pole} = \frac{1}{1-s/s_{cav}}$
exact	$F_{+,\times}(\phi,\theta)$	sinc-function	$H_{\rm FP} = \frac{1-r_a r_b}{1-r_a r_b e^{-2sT}}$
Grishchuk	$F_{+,\times}(\phi,\theta)$	sinc-function	1

Gravi-magnetic effect

from D. Baskaran and L.P. Grishchuk, *Classical and Quantum Gravity*, Vol. 21, p. 4041, 2004

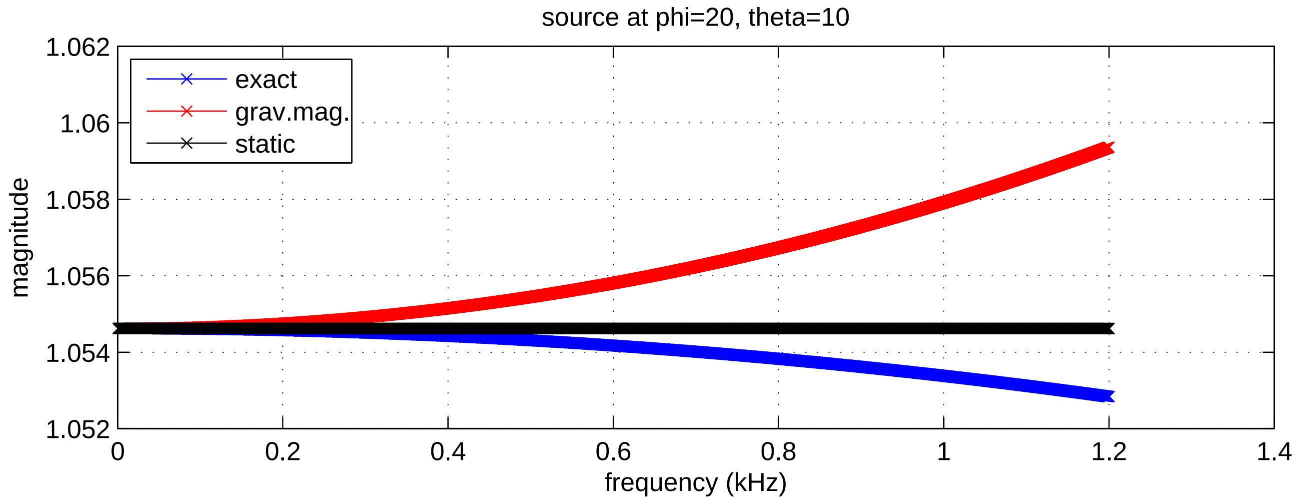
Essential steps:

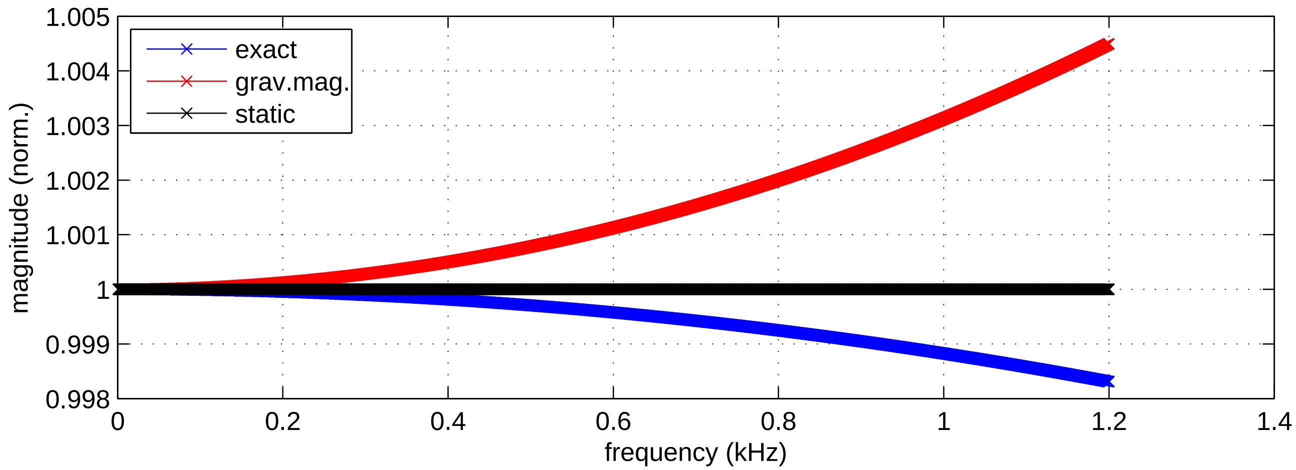
1. begin with the exact formula: $H_{gw}(s)$,
2. take out the Fabry-Perot effect. (i.e. put $H_{\mathrm{FP}}(s) = 1$),
3. expand the result in powers of f (frequency) or s,
4. keep the zeroth and the first order terms,

The zeroth order terms represent static antenna patterns $[F_{+,\times}(\Omega)]$ and are called "electric components". The first order terms represent a correction from the frequency dependence of the antenna patterns (sinc-functions) and are called "magnetic components".

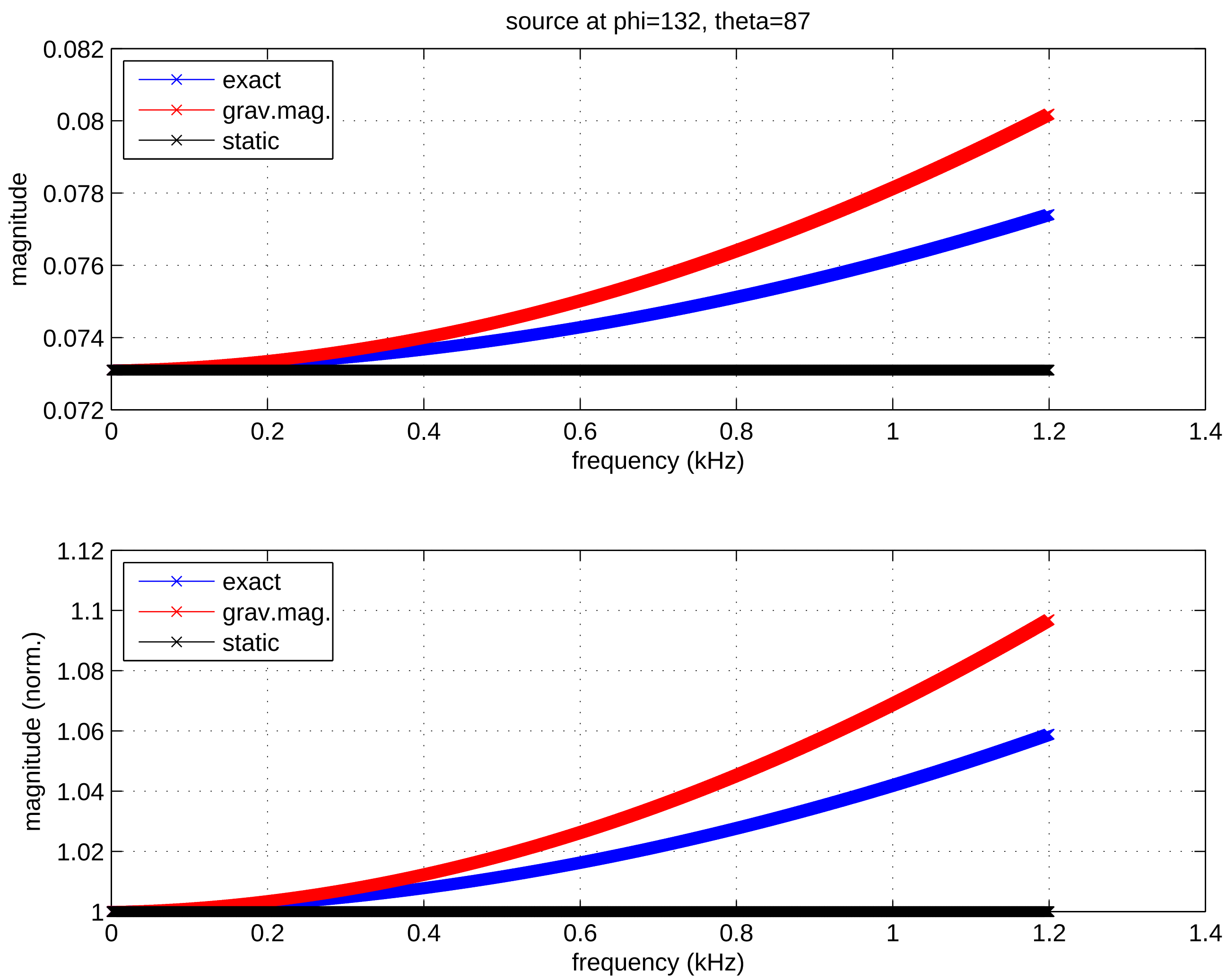
Grishchuk at CaJAGWR: "...the magnetic component provides a correction of up to 10% in the frequency band of 1200 Hz."

Take a close look.



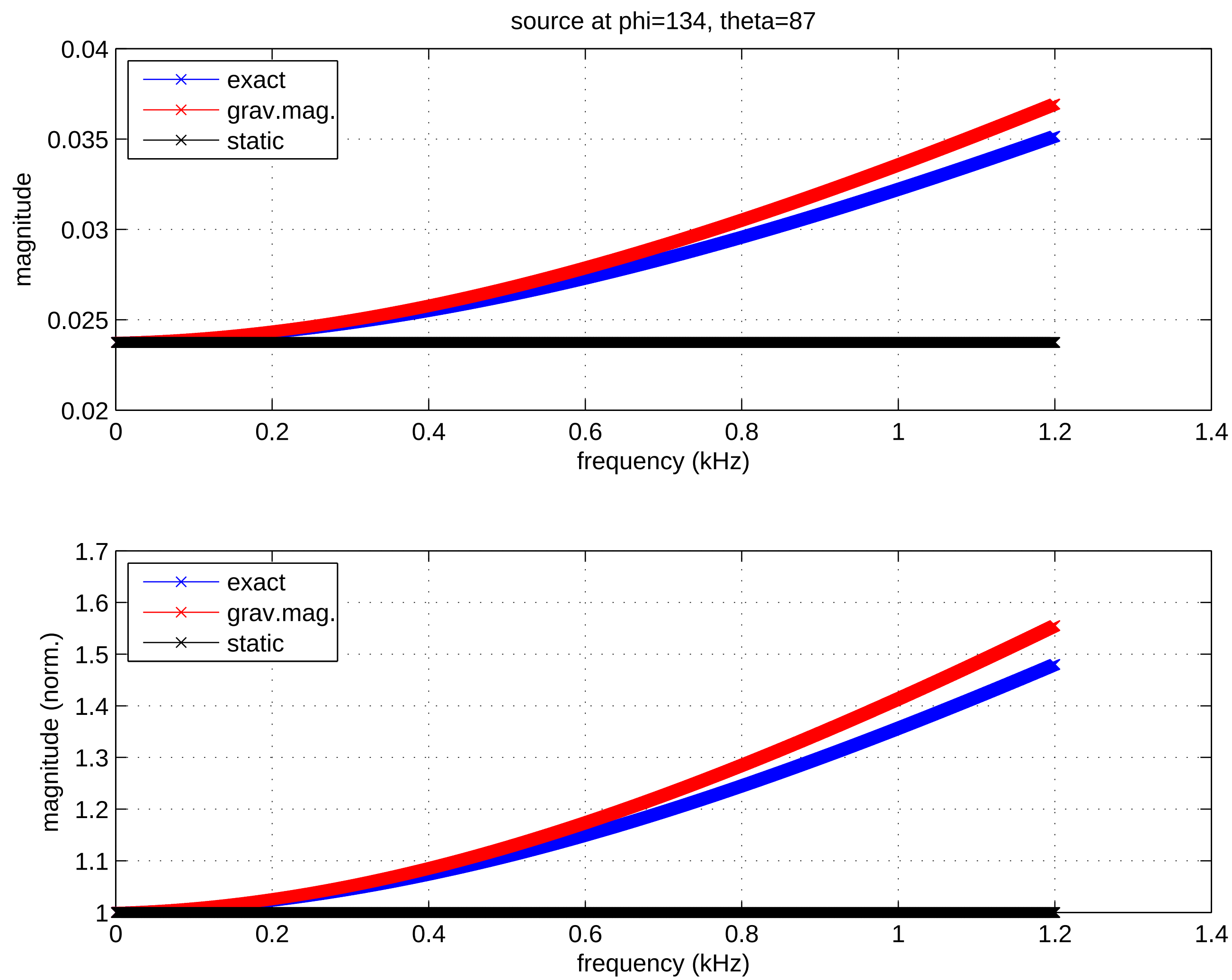


Give me ten!

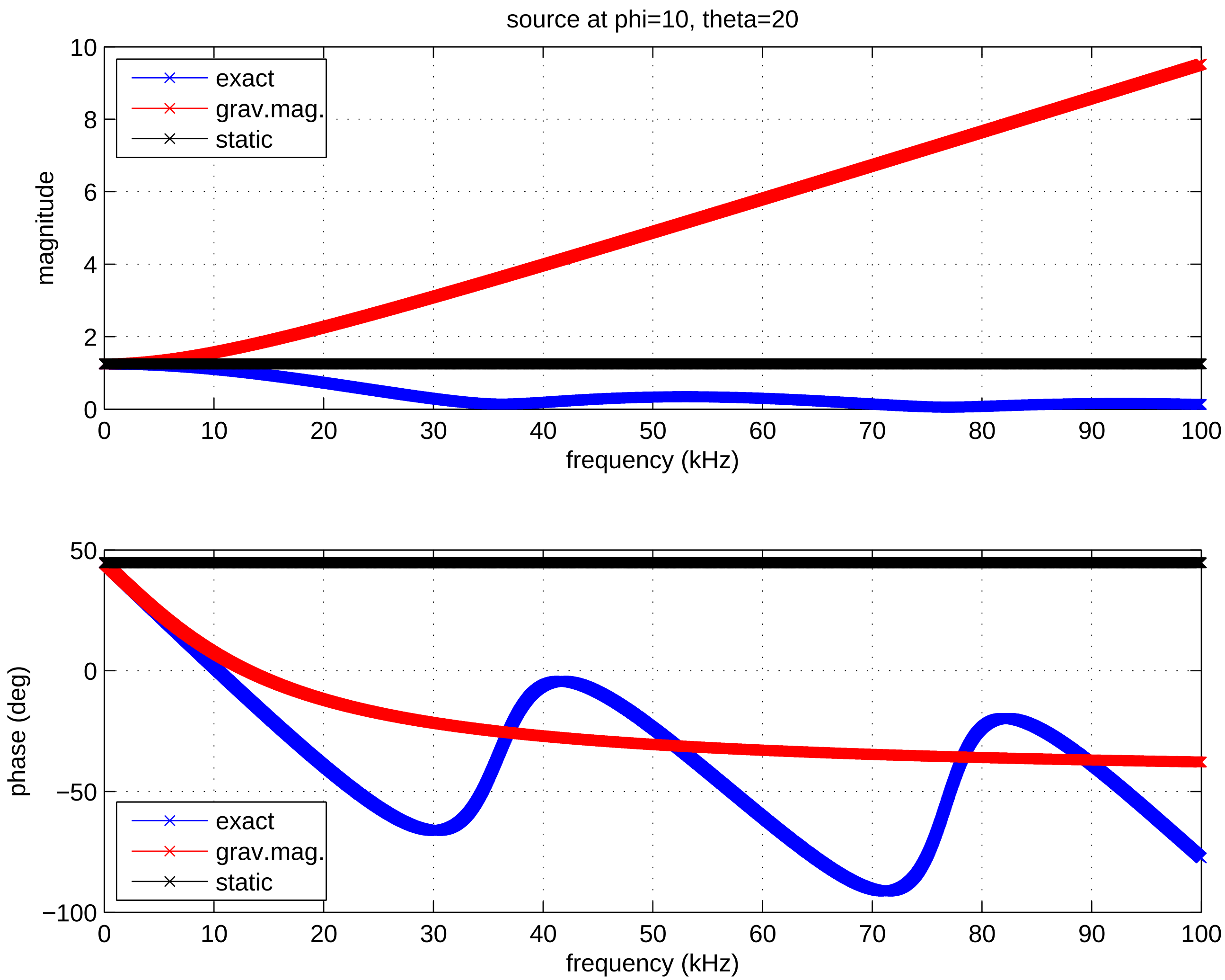


source at phi=132, theta=87
exact
grav.mag.
static
magnitude
0.082
0.08
0.078
0.076
0.074
0.072
0
0.2
0.4
0.6
0.8
1
1.2
1.4
frequency (kHz)
exact
grav.mag.
static
magnitude (norm.)
1.12
1.1
1.08
1.06
1.04
1.02
1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
frequency (kHz)

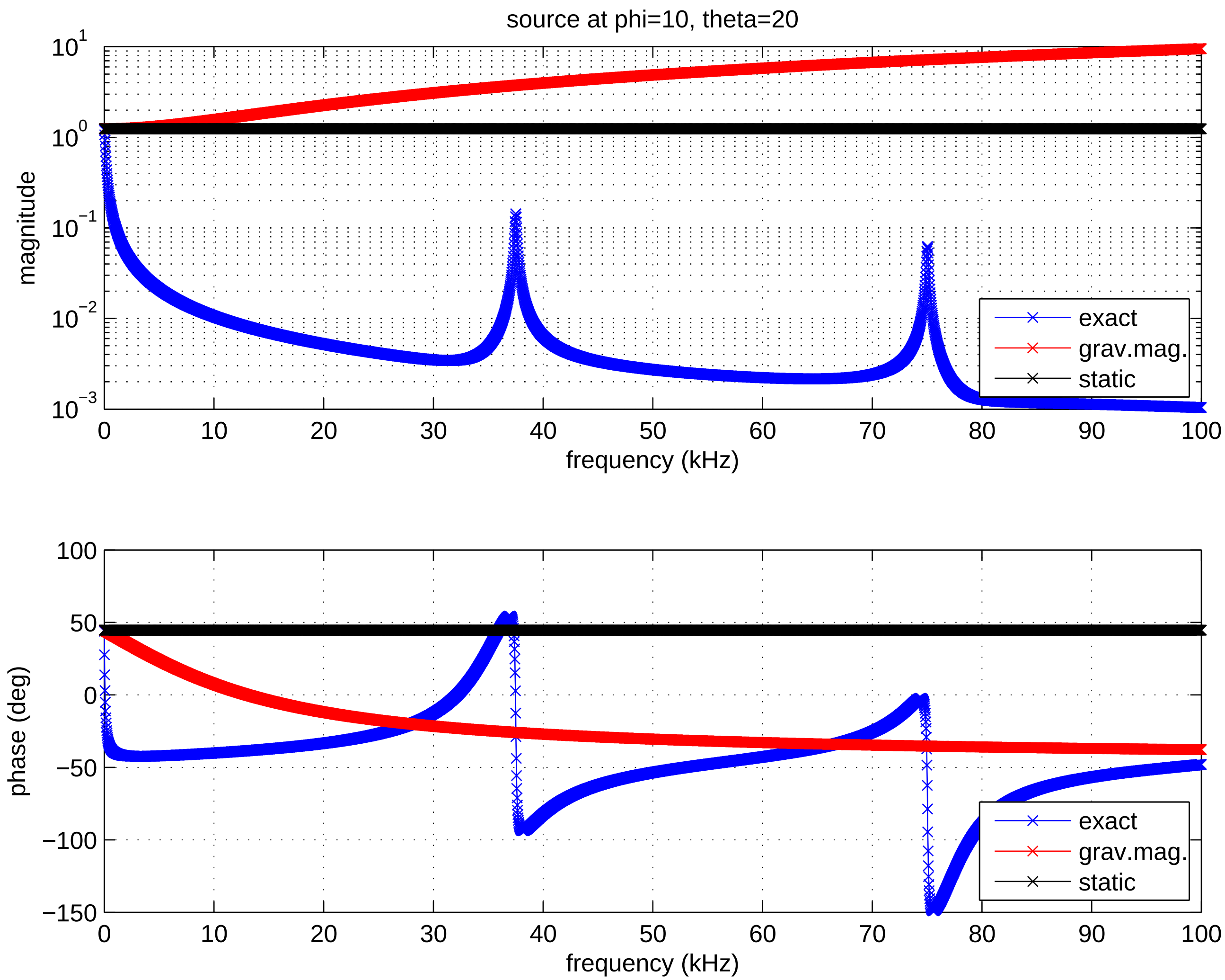
Can you give me more?



Same thing, bigger span



Put the cavity back, please!



All good things come to an end

Etimology of *Gravi-magnetic* component: It was suggested by an analogy with electro-magnetism, or the Lorentz force, because of the time derivative. But the actual coupling is $x \times dh/dt$ and not $dx/dt \times h$. From a more fundamental point of view: can gravity couple to velocity?

So "What's in a name?"

Lee Iacocca: "Everything!"

Conclusions:

- Antenna patterns are frequency dependent.
- The first order correction is good, but the exact formula is even better.
- The frequency dependence does not introduce significant corrections at frequencies in our bandwidth. (no need to change the calibration model.)
- Static antenna patterns combined with the single-pole transfer function well approximate the exact response. (two errors cancell each other!)